

Linear Algebra I

Problem Sheet 2: Linear Maps

Bases

Question 1. Complete the proof from class that states any two bases of a finitely generated vector space have the same number of elements.

Question 2. Prove that the kernel of a linear map is trivial ($\ker(L) = \{0\}$) if and only if the map L is injective. (In fact, this is true of all *homomorphisms*, something we will see in Abstract Algebra.)

Question 3. Argue whether the following maps L between vector spaces can be linear.

- Say $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ whose range is a circle in the plane;
- Let $a = (1, 0, 0, 0)$, $b = (1, 1, 0, 0)$, and $c = (1, 1, 1, 0)$. Say $L : \mathbb{R}^3 \rightarrow \mathbb{R}^6$ such that $L(a) = L(b) = L(c) = 0$ and $L(3, -19, 7) = (1, 1, 1, -3, 6, 2)$.

Rank and determinant

Question 4. Construct a matrix that has full rank but 0 determinant.

Definition. Consider the following operation on vectors in $\mathbb{R}^2$:

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c & d \end{pmatrix} = \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}.$$

Any matrix that is a product of vectors in this way is called *elementary*.

Question 5. Show that a matrix has a two dimensional columnspace if and only if it can only be written as a sum of two non zero elementary matrices. (Here non zero means at least one entry is non zero).